

Dissipation induced non-Gaussian energy fluctuations

ERIC BERTIN AND PETER C. W. HOLDSWORTH

Université de Lyon, Laboratoire de Physique, École normale supérieure de Lyon, CNRS, 46 allée d'Italie, F-69007 Lyon, France

PACS 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion

PACS 02.50.Ey – Stochastic processes

PACS 47.27.eb – Statistical theories and models

Abstract. – The influence of dissipation on the fluctuation statistics of the total energy is investigated through both a phenomenological and a stochastic model for dissipative energy-transfer through a cascade of states. In equilibrium the states obey equipartition and the total energy obeys the central limit theorem, giving Gaussian fluctuation. In the presence of dissipation the fluctuations can be driven non-Gaussian if there is macroscopic energy transfer from large to small scales. We are thus able to equate the non-Gaussian order parameter fluctuations in model equilibrium systems at criticality with energy fluctuations in these dissipative systems. Energy fluctuations in the phenomenological model map directly onto the $1/f^\alpha$ -noise problem and numerical simulations of the stochastic model yield results in qualitative agreement with these predictions.

Introduction. – Dissipative many body systems such as turbulent flows, granular matter or evolving dry foams can be parameterized by spatially averaged energy functions for the integrated kinetic energy, energy dissipation, or injection in the case of driven systems. Despite the many body nature of these systems, the probability density for such global quantities is often found to be non-Gaussian (for example see [1–13]), with finite skewness and a greater probability for large deviations than predicted by the central limit theorem. Although the skewness varies from one system to another, a generic pattern does seem to emerge [14] in which the energy distribution can often be approximated by a generalized Gumbel function [15–18] – a distribution usually appearing in the context of extreme value statistics [19,20]. Such functions also parameterize critical order parameter fluctuations in a class of Gaussian models sitting at their lower critical dimension. In particular they give the form of order parameter fluctuations along the line of critical points of the 2D-XY model to an excellent approximation [14–16], as well as the form of one-dimensional interface fluctuations showing $1/f$ noise spectra [21,22]. In these equilibrium systems, while the probability for order parameter fluctuations is non-Gaussian, energy fluctuations perfectly satisfy the central limit theorem, as the energy statistics for individual elements is given by equipartition. Moving the non-Gaussian fluctuations from order parameter to energy suggests a breakdown of equipartition and the absence of

a unique effective temperature covering all scales of the system. This is the subject of this paper. In the next section, using a simple model of a dissipative system, with energy transfer between scales and scale dependent dissipation, we show how dissipation induces a non-uniform energy profile among scales and drives the system towards a steady state with scale dependent temperature. In the third section, we test these ideas against a simple stochastic cascade model which we simulate numerically. Finally, in the discussion, we argue that for a dissipative systems with macroscopic energy transfer, the concept of equipartition of energy at equilibrium should be replaced by an equipartition of energy transfer in the steady state.

A generic phenomenological model. – In the following we construct a simple model which develops the above phenomenology of non-Gaussian energy fluctuations and spatial correlations when driven out of equilibrium. The model consists of a large number of Fourier modes ϕ_q , with q a wave vector, such that the energy associated with mode q is $\varepsilon_q = \frac{1}{2}|\phi_q|^2$, considered as a random variable. The modes with different q values are considered to be statistically independent. In the spirit of a local (in scale) equilibrium approximation, we consider that all distributions $p_q(\varepsilon_q)$ have an equilibrium form, with a q -dependent temperature T_q ,

$$p_q(\varepsilon_q) = \frac{1}{T_q} e^{-\varepsilon_q/T_q}, \quad (1)$$

a scenario which has been shown to be exact in a cascade model similar in spirit to the one considered here [23]. At equilibrium, one has $T_q = T_0$ for all q , so that the total energy $E = \sum_q \varepsilon_q$ is simply a sum of independent and identically distributed variables. From the central limit theorem, it follows that the distribution of the total energy is Gaussian in the limit of a large number of modes. Dissipative fluid systems, such as turbulent flows and granular gases can be modeled at this level by considering an energy cascade process from large to small length scale, with dissipation occurring principally at small scales so that the cascade terminates at the dissipative scale q_0 . Note that phenomenologically, a one-dimensional cascade is often considered. In this case however, one should take into account the fact that the modes are space filling in d dimensions, so that each level of the cascade consists of a shell of order q^{d-1} modes ϕ_q . We shall come back to this point in the discussion. For the present, we do not take this density of state issue into account, and consider a purely one-dimensional cascade. For simplicity, we assume that energy transfer is diffusive, driven by a temperature gradient between scales, so that the diffusive energy flux is given by

$$J_{\text{dif}}(q) = -D(q) \frac{\partial T}{\partial q} \quad (2)$$

where $D(q)$ is a scale-dependent diffusion coefficient. Note that we have implicitly replaced the discrete set of temperatures T_q by a function $T(q)$ of the continuous variable q defined on the interval $1 \leq q \leq q_0$. In order to model a scale invariant system, it is natural to choose a power-law form,

$$D(q) = D_0 q^{1+\alpha}. \quad (3)$$

For $\alpha = 0$, the diffusive flux reduces to $J_{\text{dif}}(q) = -\partial T / \partial \ln q$, so that the dynamics is exactly the same at all scales. The parameter α thus quantifies the dynamical bias between large and small scales. Injection takes place at $q = 1$, and we further assume that dissipation is localized at $q = q_0 \gg 1$, so that energy is locally conserved in the interval $1 < q < q_0$. The conservation equation reads

$$\frac{\partial T}{\partial t} = -\frac{\partial}{\partial q} J_{\text{dif}}, \quad (4)$$

leading to the following diffusive evolution equation for the temperature $T(q)$:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial q} \left(D(q) \frac{\partial T}{\partial q} \right). \quad (5)$$

This equation is supplemented by the boundary conditions at $q = 1$ and at $q = q_0$. At $q = 1$ the injection mechanism is modeled by assuming that it fixes the average value of the large scale energy: $T(q = 1) = T_0$. At $q = q_0$, the flux of dissipated energy

$$J_{\text{dis}} = \nu T(q_0), \quad (6)$$

is equal to the diffusive flux $J_{\text{dif}}(q_0)$. The dissipation coefficient ν is assumed to be much larger than the diffusion

coefficient $D(q_0)$, consistently with the assumption that dissipation is localized at the scale q_0 .

We now focus on the stationary state of the model; $\partial T / \partial t = 0$, so that $J_{\text{dif}}(q) = J_0$, independently of scale. This condition enforces a temperature gradient between scales in the out-of-equilibrium steady state, thus ensuring that the system cannot support a single, effective temperature when driven away from equilibrium. It follows that

$$\frac{\partial T}{\partial q} = -\frac{J_0}{D_0 q^{1+\alpha}}. \quad (7)$$

Taking into account the boundary condition $T(q = 1) = T_0$, the temperature profile is given after integration by

$$T(q) = \left(T_0 - \frac{J_0}{\alpha D_0} \right) + \frac{J_0}{\alpha D_0} q^{-\alpha}. \quad (8)$$

Using the equality of diffusive and dissipative fluxes at $q = q_0$, we can determine J_0 , yielding

$$J_0 = T_0 \left[\frac{1}{\nu} + \frac{1 - q_0^{-\alpha}}{\alpha D_0} \right]^{-1}. \quad (9)$$

The resulting value of the flux thus depends on α . If $\alpha > 0$, given that $q_0 \gg 1$ and $D_0/\nu \ll 1$, a finite flux J_0 is obtained,

$$J_0 \approx \alpha D_0 T_0, \quad (10)$$

while for $\alpha < 0$, the flux becomes very small

$$J_0 \approx \frac{|\alpha| D_0 T_0}{q_0^{|\alpha|}}. \quad (11)$$

This transition between two regimes as a function of the parameter α is similar to that reported in [23]. From Eq. (8), one finds that the temperature profile $T(q)$ behaves for $\alpha > 0$ as a power law of q ,

$$T(q) \approx \frac{T_0}{q^\alpha}. \quad (12)$$

In contrast, for $\alpha < 0$, the temperature profile is

$$T(q) \approx T_0 \left[1 - \left(\frac{q}{q_0} \right)^{|\alpha|} \right] \quad (13)$$

and is thus essentially constant, and equal to T_0 , unless q is close to q_0 (on a logarithmic scale). Note that these asymptotic results rely on $q_0^{|\alpha|}$ being large. For small α , one thus expects strong finite size effects, as very large values of q_0 may be needed to fulfill $q_0^{|\alpha|} \gg 1$. The temperature gradient is therefore seen to be intimately related to finite energy transfer down the scales and hence to energy dissipation at small scale. For negative values of α , although the transfer becomes small for large q_0 , it remains sufficient to establish a local equilibrium on a sufficiently long time scale.

We now turn to the fluctuations of the total energy $E = \sum_q \varepsilon_q$, where ε_q is distributed according to Eq. (1),

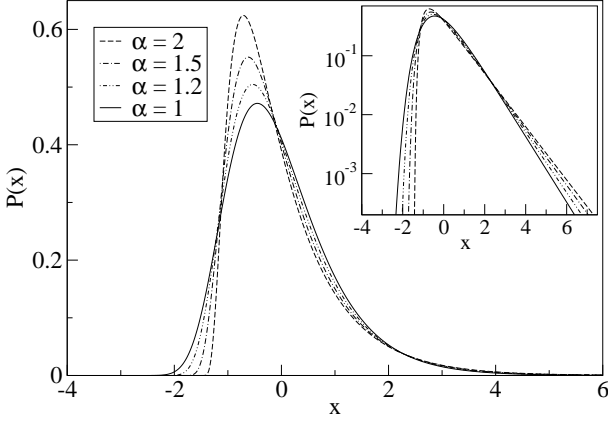


Fig. 1: Illustration of the $1/f^\alpha$ -noise distributions for several values of α . Inset: same distributions on a semi-logarithmic scale.

with the temperature T_q satisfying either Eq. (12) or Eq. (13). It is clear that the energy E is a sum of independent random variables with non-identical distributions. If $\alpha < 0$, the distributions of ε_q are almost the same for all q , so that the central limit theorem holds in the limit $q_0 \rightarrow \infty$. In the opposite case $\alpha > 0$, the distribution of ε_q reads

$$p_q(\varepsilon_q) = \kappa q^\alpha e^{-\kappa q^\alpha \varepsilon_q}, \quad (14)$$

with $\kappa = 1/T_0$, and one recovers the so-called $1/f^\alpha$ -noise problem investigated in [22]. To determine the shape of the energy distribution, we introduce the rescaled energy x defined as

$$x = \frac{E - \langle E \rangle}{\sigma_E}, \quad (15)$$

where $\langle E \rangle$ and σ_E are the empirical mean values and standard deviations of the energy E in each case.

One might expect non-Gaussian statistics to appear for all $\alpha > 0$. However, it turns out that the distribution $P(x)$ remains Gaussian in the large N limit as long as $\alpha \leq \frac{1}{2}$ [22]. For $\alpha > \frac{1}{2}$, the limit distribution is non-Gaussian, and becomes more and more asymmetric with increasing α , crossing over from Gaussian for $\alpha \rightarrow \frac{1}{2}^+$ to the exponential distribution for $\alpha \rightarrow \infty$. For $\alpha = 1$, one obtains the Gumbel distribution [21, 24], originally known in the context of extreme value statistics [19, 20]. Remarkably, it seems that experimental and numerical systems often present results that can be described to a good approximation by a generalized Gumbel distribution close to this form [1–9]. The non-Gaussian distributions are illustrated in Fig. 1. From the results of [22], one easily obtains the skewness and kurtosis of these distributions as

$$\langle x^3 \rangle = \frac{2\zeta(3\alpha)}{\zeta(2\alpha)^{3/2}} \quad (16)$$

$$\langle x^4 \rangle - 3 = \frac{6\zeta(4\alpha)}{\zeta(2\alpha)^2} \quad (17)$$

where ζ is the Riemann Zeta function, defined as $\zeta(a) = \sum_{n=1}^{\infty} n^{-a}$. Note that by definition $\langle x \rangle = 0$ and $\langle x^2 \rangle =$

1. Both the flatness and the kurtosis go to zero when $\alpha \rightarrow \frac{1}{2}^+$, consistently with the fact that the $1/f^\alpha$ -noise distributions converge to the Gaussian law in this limit.

To sum up, the main outcome of the generic phenomenological model considered in this section is a transition between Gaussian and non-Gaussian energy distributions, occurring when the parameter α characterizing the transfer properties between scales is varied. For a finite number of modes, the transition should be replaced by a crossover between close-to-Gaussian and strongly non-Gaussian distributions. It is plausible that the precise crossover value of α may depend on the details of the model, but one may expect the scenario to be qualitatively valid in more generic situations. In the next section, we test this scenario in an explicit stochastic model.

Stochastic dissipative model. — As a test of the above scenario, we consider a dissipative energy transfer model explicitly defined by stochastic rules. This model is similar in spirit to the ones considered in [17, 23, 25], though some significant differences are present. The energies are positive real variables ε_q attached to the wavenumbers $q = 1, \dots, q_{\max}$, forming a one-dimensional lattice with an energy cascade from low to high wavenumber. As in the phenomenological model, the energy of mode q is $\varepsilon_q = \frac{1}{2}|\phi_q|^2$, where ϕ_q is a Fourier amplitude. The dynamics is defined by three different mechanisms: energy injection at large scale, internal energy transfer between neighboring scales and local energy dissipation. To preserve scale invariance, the presence of energy transfer between wavenumbers q and q' is determined according to the ratio q'/q ; namely it occurs only if $1/\lambda < q'/q < \lambda$, where $\lambda > 1$ is a fixed parameter. The different transfer rates are defined as follows: an amount of energy μ is either injected at $q = 1$ with rate (probability per unit time) $I(\mu)$, transferred from a site q to site q' with a rate $\Gamma_{q,q'}(\mu|\varepsilon_q)$, or dissipated from site q at a rate $\Delta_q(\mu|\varepsilon_q)$. Note that the transfer and dissipation rates depend on the energy ε_q present at wavenumber q . We choose the following forms for the rates

$$I(\mu) = T_0^{-1} e^{-\mu/T_0}, \quad (18)$$

$$\Gamma_{q,q'}(\mu|\varepsilon_q) = D_0(qq')^{\alpha/2} \varepsilon_q, \quad (19)$$

$$\Delta_q(\mu|\varepsilon_q) = \nu_0 q^2 \varepsilon_q. \quad (20)$$

Choosing this form for $I(\mu)$ means that energy is injected at wavenumber $q = 1$ by connecting this element to a thermostat at temperature $T_0 = 1$, which fixes the energy scale. Energy is also transferred back to the reservoir with a rate $\tilde{I}(\mu|\varepsilon_q) = \varepsilon_q$. The value of the parameter α is again the key to the way in which energy is transferred between scales. The constant D_0 in Eq. (19) is chosen such that $D_0(qq')^{\alpha/2}$ remains bounded when $q_{\max} \rightarrow \infty$. We take $D_0 = 1$ for $\alpha \leq 0$ and $D_0 = q_{\max}^{-\alpha}$ for $\alpha > 0$. Finally, the constant ν_0 controls the dissipation rate and can be interpreted as a viscosity. In the following, we shall be interested in the limit where ν_0 takes small, but non-zero

values. Note that the dissipation rate is the only dynamical rule that breaks detailed balance. In the absence of dissipation ($\nu_0 = 0$) the system converges to the equilibrium Gibbs distribution with temperature $T_q = T_0$ at all scales. We shall see however that the limit $\nu_0 \rightarrow 0$ is singular, as is well-known in the context of fluid turbulence [26].

The main differences between the present cascade model and that introduced in [17] are that in the latter the energy transfer rate is fully biased and scale-independent and that the dissipation rate has an ad hoc form chosen to make the model analytically solvable. Here, in addition to the inclusion of back transfer from large to small scale, we choose a dissipation rate proportional to q^2 , which is natural in the context of viscous damping.

With this form, as long as $\alpha < 2$, the ratio of the dissipated to transferred energy increases as the wavenumber q increases, assuring a cut-off wavenumber for the cascade:

$$q_0 \sim \nu_0^{1/(\alpha-2)} D_0^{\alpha/(\alpha-2)}. \quad (21)$$

In the context of simulations of a finite system with a number q_{\max} of modes, the characteristics of the cascade are determined by both q_0 and q_{\max} and their interplay, as we show below.

We have simulated the model for $q_{\max} = 500$, and measured the steady-state distribution of the total energy for several values of α . The resulting distributions $P(x)$ of the rescaled energy x are shown for $\nu_0 = 10^{-5}$ in the upper panel Fig. 2, on a linear scale in the main figure and log-linear scale in the inset. For $\alpha = 2$ the shape is strongly non-Gaussian, with a clear skewness and long tail for fluctuations for positive values of x . As α decreases from 2 to 1 there is a sharp decrease in the anisotropy of the distribution in qualitative agreement with the data for the phenomenological model shown in Fig. 1. The form for $\alpha = 1$ is close to that typically observed in experiments and simulations for dissipative systems (for example see [1–9]). Further reduction in the value of α sees a slow evolution towards a Gaussian distribution although it is not possible to determine clearly from this data if a change in regime occurs for α around zero.

Evidence for such a crossover is however given by the evolution of the energy spectrum $\langle \varepsilon_q \rangle$ with α , shown in the lower panel of Fig. 2. There is a clear quantitative difference between the spectra for positive and negative values of α . For $\alpha = 2$, the spectrum is approximately power law over five orders of magnitude in energy (with an exponent $\tilde{\alpha} \approx -2.9$), with no evidence of a crossover to a dissipative regime, which is consistent with the ratio of energy transfer to dissipation being scale independent. For $\alpha = 1$ the inertial range of approximate power law scaling (here with an exponent $\tilde{\alpha} \approx -1.6$) is cut off by a build up of energy at a scale around $q = 2 \times 10^2$, which corresponds to the dissipation scale, q_0 for this value of ν_0 . One might have expected a fall off in energy per mode as one goes to larger wavenumber. However, in this case

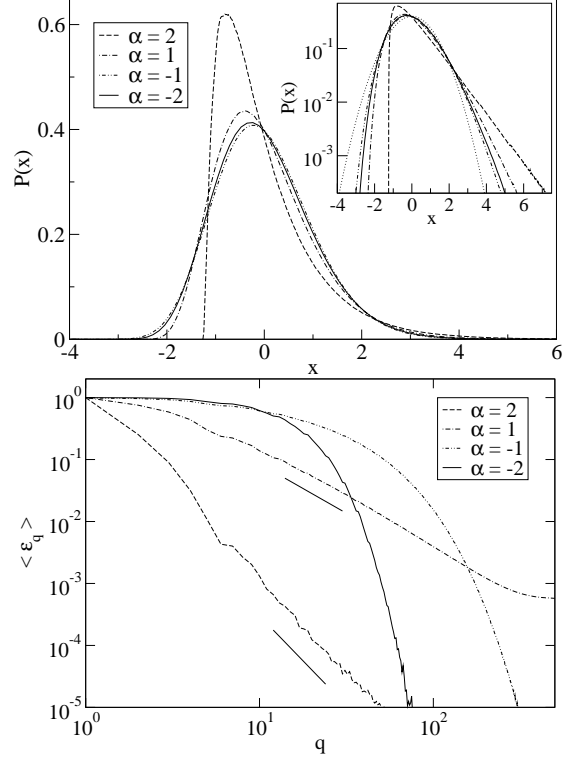


Fig. 2: Top panel: Distribution of the rescaled energy $x = (E - \langle E \rangle) / \sigma_E$ in the stochastic cascade model for different values of the α characterizing energy transfer through scales ($\nu_0 = 10^{-5}$). Inset: same data on a semi-logarithmic scale. The dotted line is the Gaussian distribution, shown for comparison. Bottom panel: energy spectrum $\langle \varepsilon_q \rangle$ for the same values of α , on a log-log scale. Straight lines indicate slopes of -2.9 ($\alpha = 2$) and -1.6 ($\alpha = 1$).

q_0 is close to q_{\max} with the result that energy is stored at the bottom of the cascade, rather than being dissipated at larger wavenumber. Note that when simulations are run for smaller values of ν_0 , we find that the measured exponents $\tilde{\alpha}$ decrease in magnitude (data not shown) towards α , as for the phenomenological model presented in the previous section. More extensive finite size scaling is clearly necessary here in order to capture the details of the cascade and make a quantitative comparison between the phenomenological and the numerical model, but the general behavior of extensive energy transfer, inertial scaling regime and non-Gaussian energy fluctuations appears well established. For negative values of α the spectrum again mirrors that for the phenomenological model, being constant at small q and cut off to zero around the characteristic dissipation scale, $q_0 \sim 50$ for $\alpha = -1$ and $q_0 \sim 15$ for $\alpha = -2$.

The deviation from the Gaussian distribution for $\alpha \lesssim 0$ can be traced to finite viscosity effects. We have computed the distribution $P(x)$ for decreasing values of the viscosity ν_0 for $\alpha = -2$, shown in Fig. 3. As in this case we estimate the crossover scale $q_0 \sim \nu_0^{-1/4}$, we should be able to push ν_0 as low as 10^{-8} before the maximum cut off

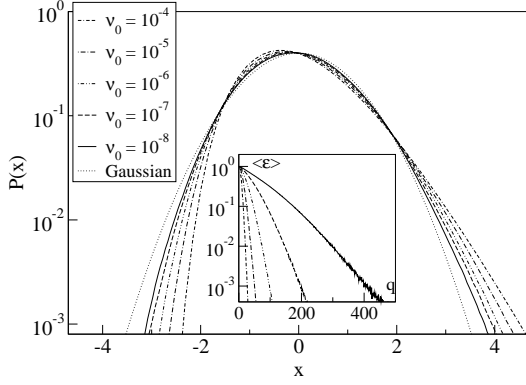


Fig. 3: Distribution $P(x)$ of the rescaled energy x for decreasing values of the viscosity ν_0 ($\alpha = -2$). A slow convergence to the Gaussian distribution (dotted line) is observed. Inset: average energy $\langle \epsilon_q \rangle$ as a function of q for the same values of ν_0 (ν_0 decreases from bottom to top).

$q_{\max} = 500$ plays a significant role. A slow convergence to the Gaussian distribution is observed, as ν_0 decreases. The convergence is indeed associated with the increase of the characteristic decay scale of the energy spectrum $\langle \epsilon_q \rangle$ when $\nu_0 \rightarrow 0$, q_0 (see inset of Fig. 3), which does indeed scale roughly as $\nu_0^{-1/4}$ multiplied by a factor of order unity. This evolution is consistent with an increase in the number of degrees of freedom which are equilibrated at temperature T_0 , allowing an approach to Gaussian behavior for $P(x)$ as ν_0 goes to zero.

To provide further evidence for a crossover from Gaussian to non-Gaussian behavior as a function of α , we compute the skewness $\langle x^3 \rangle$ and the kurtosis $\langle x^4 \rangle - 3$, as a function of α for fixed $\nu_0 = 10^{-5}$. Results are plotted in the Fig. 4. The skewness and kurtosis take small values for $\alpha \lesssim 0.5$, and larger values for $\alpha \gtrsim 0.5$. The data can be compared with the analytic predictions from the phenomenological model. Although we are not able to make a quantitative statement, the simulation results are clearly consistent with the phenomenology within the accuracy achieved. For $\alpha \lesssim 0$, the skewness and kurtosis can be made smaller by decreasing the viscosity ν_0 , as shown in Fig. 3. In contrast, for positive values of α , numerical simulations are difficult for very low viscosities, as q_{\max} must exceed q_0 to avoid energy accumulation at large wavenumber. A more detailed study therefore requires finite size scaling both in viscosity and q_{\max} which we leave for a future study. However the results presented in this section show that the scenario of an energy transfer driven crossover between near-Gaussian and strongly non-Gaussian energy fluctuations is also present in this stochastic cascade model.

Discussion. — In the previous sections we have shown that the asymmetry of the energy distribution in dissipative systems can be traced back to the efficiency of the transfer mechanism from large to small scales. This departure from the central limit theorem has, in the past

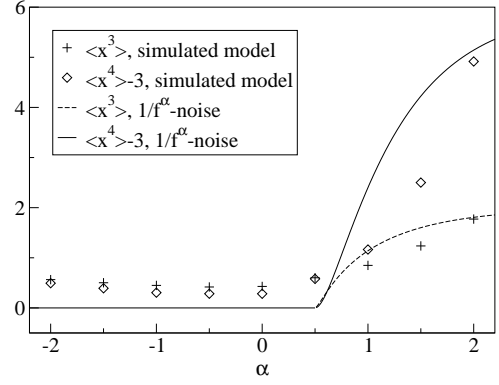


Fig. 4: Skewness and kurtosis as a function of α for the simulated model (symbols) and for the $1/f^\alpha$ -noise distributions (lines), with a viscosity $\nu_0 = 10^{-5}$.

been compared with non-Gaussian order parameter fluctuations in model systems of criticality at equilibrium [14, 16, 21]. The asymmetry can here be understood as resulting from the constraint imposed by the equipartition of energy. Consider the Fourier amplitude of the order parameter for wave vector \mathbf{q} , $\psi_{\mathbf{q}}$ in the specific example of the low temperature phase of the 2D-XY model: the energy associated with the mode reads $\epsilon_{\mathbf{q}} = \frac{1}{2} q^2 |\psi_{\mathbf{q}}|^2$, while the contribution of wavevector \mathbf{q} to the order parameter is $m_{\mathbf{q}} = \frac{1}{2} |\psi_{\mathbf{q}}|^2$ [16]. The equipartition of energy at equilibrium imposes that the average energy $\langle \epsilon_{\mathbf{q}} \rangle$ of mode \mathbf{q} is independent of \mathbf{q} , namely $\langle \epsilon_{\mathbf{q}} \rangle = k_B T$. The equipartition thus imposes $\langle m_{\mathbf{q}} \rangle = k_B T / q^2$, so that the scale of contributions to the order parameter over the Brillouin zone diverges in the thermodynamic limit. Combining the $1/q^2$ scaling for $m_{\mathbf{q}}$ with the density of states, $g(q) \sim q$ results in non-Gaussian fluctuations for the order parameter, while the fluctuations of the total energy are perfectly Gaussian, being the sum of independent and identically distributed random variables. Similar behavior occurs in one dimension if interactions are long ranged [27], leading directly to a “ $1/f$ -noise” spectrum [21], which can in turn be generalized to the $1/f^\alpha$ behaviour [22] discussed above. This is of course an extremely simplified vision of criticality [28]: critical systems are certainly not in general integrable and non-linear coupling between modes can drive a divergent specific heat and consequently non-Gaussian energy fluctuations. However, the specific heat exponent α is systematically smaller than that for the susceptibility, γ , indicating a weaker divergence and showing that energy fluctuations are in general more constrained than those for the order parameter.

Here we consider dissipative systems at the same level of approximation, finding clear violation of the central limit theorem and non-Gaussian statistics for the spatially averaged internal energy. In the context of an energy cascade, with dissipation localized at small length scales, equipartition is replaced, in steady state, by a new constraint; the average energy transfer between scales is uniform through-

out the cascade, $J_q = J_0$. For the models with diffusive dynamics that we have considered, this steady state condition ensures that global equipartition of energy must be violated, thus introducing a scale dependent temperature, $T_q \sim q^{-\alpha}$ and non-Gaussian energy fluctuations, if α is large enough. Indeed, as $\epsilon_q = |\phi_q|^2$, one finds for both the phenomenological and stochastic models that $J_q \sim q^\alpha |\phi_q|^2$ in analogy with the energy of the equilibrium system. Hence we find quantitative equivalence between order parameter fluctuations in the equilibrium models and those for internal energy in the case of dissipation. The sum over all instantaneous energy transfers would thus satisfy the central limit theorem and this rather non-standard global quantity would have Gaussian statistics. There is once again a barrage of simplifications separating our phenomenology from real systems, but our arguments do lay out a general framework for why energy fluctuations are more violent in dissipative systems than in equilibrium and in what context one can expect energy fluctuations to be approximated by $1/f^\alpha$ -like distributions. In particular examples, such as kinetic energy fluctuations of a granular gas in a homogeneous cooling state [10], or of decaying Burgers turbulence [7], the agreement between simulation results and the model distributions is close enough to suggest that the phenomenology presented here becomes quantitatively correct, at least in these specific cases.

The comparison with equilibrium systems can be pushed further by considering the role of spatial dimension, d in the presence of dissipation. At a heuristic level, a naive reasoning involves treating modes at each wave vector as statistically independent and consists in simply multiplying the $1/f^\alpha$ spectrum by the density of states $g(q) \sim q^{d-1}$, yielding an effective exponent $\alpha' = \alpha - (d-1)$ for the energy spectrum. If α is large enough, a transition from non-Gaussian to Gaussian fluctuations is observed when increasing d beyond $d_u = \alpha + \frac{1}{2}$ (corresponding to $\alpha' = \frac{1}{2}$), which can be interpreted as an upper critical dimension, in qualitative analogy with equilibrium critical phenomena. Taking into account correlations between the mode at each level of the cascade will however reduce the number of effective degree of freedom, probably placing the effective exponent in the range, $\alpha - (d-1) < \alpha' < \alpha$.

In conclusion, our main results are two-fold. Firstly, we have shown that there generically exists a crossover from Gaussian to strongly non-Gaussian fluctuations of the spatially averaged energy, when varying the transfer properties through scales (quantified here by the parameter α) in dissipative systems. Secondly, we have argued that in the non-Gaussian regime, the asymmetry of the energy distribution is driven by a finite energy flux crossing the system from injection to dissipative scales. The constancy of this flux across scales plays a role similar to that of energy equipartition. In consequence non-Gaussian energy fluctuations arise in complete analogy to those for the order parameter at equilibrium. As for future work, it would be of interest to try to quantitatively validate this scenario in more realistic models (shell models, Burgers

turbulence, granular gases...).

Acknowledgments. – It is a pleasure to thank Steven T Bramwell, Maxime Clusel and Olivier Dauchot for useful discussions and related collaborations.

REFERENCES

- [1] R. Labbé, J.-F. Pinton, S. Fauve, J. Phys. II (France) **6**, 1099 (1996).
- [2] J.-F. Pinton, P.C.W. Holdsworth and Raul Labbé, Phys. Rev. E. **60**, R2452, (1999).
- [3] Tibor Tóth-Katona and J. T. Gleeson, Phys. Rev. Lett. **91**, 264501, (2003).
- [4] B. Ph. van Milligen, R. Sánchez, B. A. Carreras, V. E. Lynch, B. LaBombard, M. A. Pedrosa, C. Hidalgo, B. Gonçalves, R. Balbín, Phys. Plasmas **12**, 052507 (2005).
- [5] I. Sandberg, S. Benkadda, X. Garbet, G. Ropokis, K. Hizanidis, and D. del-Castillo-Negrete Phys. Rev. Lett. **103**, 165001, (2009).
- [6] S. Aumaitre, S. Fauve, S. McNemara, P. Poggi, Eur. Phys. J. B **19**, 449 (2001).
- [7] A. Noullez, J.-F. Pinton, Eur. Phys. J. B **28**, 231 (2002).
- [8] J. J. Brey, M. I. García de Soria, P. Maynar, M. J. Ruiz-Montero, Phys. Rev. Lett. **94**, 098001 (2005).
- [9] P. Visco, A. Puglisi, A. Barrat, F. van Wijland, E. Trizac, Eur. Phys. J. B **51**, 377 (2006).
- [10] J. J. Brey, A. Domínguez, M. I. García de Soria, P. Maynar, Phys. Rev. Lett. **96**, 158002 (2006).
- [11] J. Farago, J. Stat. Phys. **107**, 781 (2002).
- [12] B. Portelli, P. C. W. Holdsworth, J.-F. Pinton, Phys. Rev. Lett. **90**, 104501 (2003).
- [13] J. Farago, E. Pitard, J. Stat. Phys. **128**, 1365 (2007).
- [14] S. T. Bramwell, P. C. W. Holdsworth, J.-F. Pinton, Nature (London) **396**, 552 (1998).
- [15] S. T. Bramwell *et al.*, Phys. Rev. Lett. **84**, 3744 (2000).
- [16] S. T. Bramwell *et al.*, Phys. Rev. E **63**, 041106 (2001).
- [17] E. Bertin, Phys. Rev. Lett. **95**, 170601 (2005).
- [18] E. Bertin, M. Clusel, J. Phys. A **39**, 7607 (2006).
- [19] E. J. Gumbel, *Statistics of Extremes* (Columbia University Press, 1958; Dover publication, 2004).
- [20] J. Galambos, *The asymptotic theory of extreme order statistics* (Wiley & Sons, 1987).
- [21] T. Antal, M. Droz, G. Györgyi, Z. Rácz, Phys. Rev. Lett. **87**, 240601 (2001).
- [22] T. Antal, M. Droz, G. Györgyi, Z. Rácz, Phys. Rev. E **65**, 046140 (2002).
- [23] E. Bertin, O. Dauchot, Phys. Rev. Lett. **102**, 160601 (2009).
- [24] M. Clusel, E. Bertin, Int. J. Mod. Phys. B **22**, 3311 (2008).
- [25] E. Bertin, J. Phys. A: Math. Gen. **39**, 1539 (2006).
- [26] See, e.g., U. Frisch, *Turbulence* (Cambridge University Press, 1995).
- [27] J. F. Joanny and P. G. de Gennes, J. Chem. Phys. **81**, 552 (1984).
- [28] M. Clusel, P.C.W. Holdsworth, J.-Y. Fortin, Europhys. Lett. **76**, 1008 (2006).